Upper and Lower Bounds for Capital and Wages*

Alberto Benítez Sánchez **
Alejandro Benítez Sánchez ***

ABSTRACT

The paper considers the division of a production system into departments I and II introducing a further division of department I into a homothetic and a complementary sector. It shows the existence of nontrivial upper and lower bounds for the capital stock and the wages that depend on this division contributing, among other results, to explain the quasilinearity of the wage-profit curve found in some empirical researches.

Keywords: capital theory, income distribution, Marx, Sraffa, wage-profit curve.
JEL classification: B2, B5, P1.

RESUMEN

El artículo estudia la división de un sistema de producción en los departamentos I y II e introduce una división posterior del departamento I en un sector homotético y otro complementario. Muestra la existencia de cotas superiores e inferiores no triviales para el acervo de capital y para el salario que dependen de esta división, con lo cual contribuye, entre otros resultados, a explicar la casi linealidad de la curva salarios-ganancia encontrada en algunos estudios empíricos.

Palabras clave: teoría del capital, distribución del ingreso, Marx, Sraffa, curva salarios-ganancia.
Clasificación JEL: B2, B5, P1.

The division of a production system according to the final use given to the products of each enterprise, either as means of production or consumption goods, was originally introduced by Smith (1981: 288) and reformulated by Marx (1992: 471) who, like his predecessor, employed this division mainly to study the reproduction of the economic system. On the other hand, von Neuman (1945) built a homothetic production system to study economic growth in a general equilibrium model and, from a perspective inspired in Ricardo (2004), Sraffa (1960) built another system of this type to study the relations between prices and income distribution. The important analytical tools introduced by these authors have been incorporated in numerous publications but, to our knowledge, they have been employed separately despite the proximity of the themes treated in many of the researches.[1]

In this paper, we define the aggregates of production processes just mentioned within a model of single-product industries with no fixed capital in such a way that they result complementary. With this purpose, we first divide this system into departments I and II introducing a further division of department I into a homothetic sector and the rest of this department, to which we refer to as the complementary sector. A division of production into three parts is thus established with a corresponding division of labor into three quantities, represented synthetically by means of a vector $l = (l_h, l_c, l_{II})$, whose coordinates are the quantities of labor occupied in the homothetic sector, the complementary sector and department II, respectively. The vector, under certain election criterion specified in third section, is unique for each production system and constitutes a useful tool to study the capital stock and the wage measured with the net product, respectively KS and w, as functions of the profit rate (r).[2] This conclusion is based on the results that are presented along the paper in the order indicated.

In the first section, we expose a model originally published in Benítez (2009) whose most peculiar trait is to include all possible distributions of wage's payment between the start and the end of production.[3] In second section,

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[1] There is a large literature on these matters, Morishima (1973) studies the division in departments I and II and presents a bibliography on the subject. Bidard (2004) compares some aspect of the models by von Neuman and Sraffa and discusses the corresponding literature. In the last work, as well as in Pasinetti (1977) the interested reader may also find an introduction to the subject.

[2] Although there is no fixed capital in the model, we use the term "capital stock" in reference to the value of the physical capital in order to distinguish it from the total amount of capital, which includes the wages advanced.

[3] Apart from establishing some properties of the wage-profit curve that paper discusses the antecedents of the model. For this reason, we will only mention here that the importance of the

we establish the division into departments and prove that, given l and the maximum rate of profit (R), KS depends only on the proportion capital/(total product) in department I (c_1). In third section, we accomplish the further division of department I and prove that, given l and R, c_1 depends only on the proportion capital/(total product) in the complementary sector (c_c), a result that, together with the one just indicated implies that the same is valid for KS.

Moreover, in fourth section it is shown that, for each R, the possible values of KS are contained in the space limited by the curve $1/r$, the vertical straight line of abscise R and the two axes, in relation to which two important results are presented. According to the first one for each $r \in [0, R]$ and for each number pertaining to the interval $[0, 1/r]$ there is at least one production system for which $KS(r)$ is equal to that number. Consequently, as shown in sixth section, for each number pertaining to the interval $[0, 1]$ there is at least one production system where profits ($KS(r)r$) are equal to that number for any $r \in [0, R]$. Therefore, the graphics of the functions $KS(r)$ and those of the wage-profit curves not only belong respectively to the space just indicated and to the rectangle of diagonal $S(R) = [(0, 1), (R, 0)]$, something already known, but also cover these surfaces entirely, a conclusion not previously published to our knowledge. This situation gives relevance to the second result: each vector l determines a nontrivial lower bound for the capital stock valid for the set $C(R)$, integrated by all the production systems sharing the same R. On its turn, this bound implies an upper bound for the wage (w) that is presented in sixth section.

Given the last result, one may expect that l also determined an upper bound for KS valid for $C(R)$, but in fifth section, we prove that this is not the case. For this reason, in the same section we consider a family of subsets $C_m(R) \supset C(R)$ defined in such a manner that, for each $m > 0$, $C_m(R)$ contains all the elements of $C(R)$ except a part of those where the following takes place: for some $r^* \in [0, R]$, when r changes from r^* to R, the capital stock of the complementary sector diminishes in an amount greater than or equal to m times the simultaneous increase in its profits. The definition is helpful in the study of those production systems where, for empirical or theoretical reasons, the possibility of such a reduction in the capital stock may be excluded. In this regard, the case $m = 1$ is particularly interesting, as argued in sixth section Also in this section, we prove

payment date for income distribution is pointed out in Chapter 6 of Marx (1990: 278) and also that Sraffa (1960: 10) considers the two payment dates schedule as the most appropriate way to treat wages, although in contraposition Steedman (1977: 21) affirms that this date has no importance at all.

that for each m, l determines an upper bound for the capital stock valid for $C_m(R)$ which, together with the lower bound already mentioned, permits one to establish an estimation of $KS(r)$ whose precision may be great for certain values of r.

On the basis of the last result, in seventh section, a lower bound for the wage is determined for each set $C_m(R)$ which, together with the upper bound already mentioned, permits one to estimate $w(r)$ with an accuracy that may be high for certain levels of m, l and r. Finally, despite the fact that no empirical applications are included in this paper, in the final remarks section it is argued that the results presented here may have empirical relevance. We shall add that the roman character appearing in some statements indicates the section of the "Appendix" where the reader may find the corresponding proof.

I. THE MODEL

The model represents a productive system integrated by n industrial branches, each producing a particular type of good labeled by an index i or j, so that $i, j = 1$, 2, …, n. I will refer to a set { $j1, j2, …, jd, …, jD$} as a D set if it contains D different goods. To simplify, I will also refer to the indexes as goods. All the production processes are simultaneous and of equal duration, the quantities of each good are measured with the amount produced of the corresponding good, and the quantities of salaries with the amount of salaries paid.[4] For each pair (i, j) of indexes, a_{ij} and l_j represent respectively the quantities of i and of salaries consumed in the j industry during the period considered to produce one unit of j; for each j they verify $l_j > 0$, $a_{ij} \geq 0$ $\forall$ i and $a_{ij} > 0$ at least for one i. If the assumption is made that each quantity of salaries pays an equal quantity of labor, l_j may also be interpreted as the quantity of labor consumed in the j th industry; however, this is not required for the purposes of this paper. For each j, the price of good j in units of salary is p_j and r is the rate of profit of the period. A fraction t of the wages is paid at the beginning of production in every industry and the rest at the end, the cost of labor in each branch j is then equal to $l_j t(1 + r) + l_j(1 - t) = l_j(1 + tr)$. In these conditions, if the rate of profit is the same in every branch, the prices and costs of production are related by the following equation system:

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[4] Following Marx (1990) and Sraffa (1960) we take as given the quantities produced as well as those that are used as means of production in the different industries. In Benítez (1995) a model is presented where: a) prices are constant, b) the profit rate is the same in all industries and c) the quantities produced and consumed are determined endogenously, taking demand into account.

$$\sum_i a_{ij} p_i (1 + r) + l_j (1 + tr) = p_j \qquad\qquad j = 1, 2, \ldots, n \qquad\qquad (1)$$

We will say that (1) is viable if in every D set the sum of the quantities of each good belonging to D that are consumed directly in the production of the goods of D is not greater than 1 and is less than 1 for at least one of the goods. Consequently, every D set verifies that $\sum_d a_{i,jd} \leq 1$ for each $i \in D$ and $\sum_d a_{i,jd} < 1$ for at least one $i \in D$. I assume that every economy considered in this work is viable, which, together with the other assumptions already made permit us to verify the following propositions for every t.

Theorem 1.I There is an interval $[0, R]$ such that: a) R is independent of t and $0 < R < +\infty$, b) for each $r \in [0, R]$, the solution of (1) is unique and strictly positive, c) $p_j(r)$ is a monotonous increasing function for every j, d) at least one price tends to infinity when r tends to R, e) for each $r \in [0, R]$, the quotient $p_i(r)/p_j(r)$ is independent of $t \; \forall \; (i, j)$.

For each i, $c_i = 1 - \sum_j a_{ij}$ represents the quantity of good i produced as surplus over the amount of the same good consumed as a means of production, as (1) is viable $c_i \geq 0 \; \forall \; i$ and $c_i > 0$ for at least one i. Summing up the n equations of (1), we obtain $\sum_j \sum_i a_{ij} p_i (1 + r) + \sum_j l_j (1 + tr) = \sum_j p_j$. Substituting $\sum_j l_j$ and $\sum_j p_j$ with their respective equivalents 1 and $\sum_j \sum_i a_{ij} p_i + \sum_j c_j p_j$ in the previous equation yields $\sum_j \sum_i a_{ij} p_i (1 + r) + (1 + tr) = \sum_j \sum_i a_{ij} p_i + \sum_j c_j p_j$ and consequently:

$$\sum_j \sum_i a_{ij} p_i r + (1 + tr) = \sum_j c_j p_j \qquad\qquad (2)$$

The first term on the left-hand side of this equation is the amount of profits obtained with the means of production and the second one that of wages together with the profits corresponding to the wages advanced. As the value of the collection of goods on the right-hand side is equal to the net income of the society, we will refer to this collection as the real income.[5] Let

a) $KS = \sum_j \sum_i a_{ij} p_i / \sum_j c_j p_j$ b) $w = 1/\sum_j c_j p_j$ and   c) $K = KS + tw$ (3)

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[5] Pasinetti (1977: 134) points out that when wages are paid at the beginning of production they are not included in the classical notion of net product. Nevertheless, independently of the schedule for the payment of wages, the value of the real income is equal to the net income.

The first function represents the capital stock (equal to the cost of the means of production), the second one the wage and the third one the capital (equal to the amount invested), each variable being measured with the real income. Using this notation we can write (2) in the following forms:

a) $KSr + w(1 + tr) = 1$ and   b) $Kr + w = 1$ (4)

Dividing every term on the right-hand side of (3.a) by a price p_b arbitrarily chosen yields $KS = \Sigma_j \Sigma_i a_{ij}(p_i/p_b)/\Sigma_j c_j(p_j/p_b)$. This expression and e) from Theorem 1 imply that KS is independent of t. To simplify, we will assume in the next four sections that $t = 0$ and its other possible values will be considered only in seventh section. Under this assumption, the real income is equal to net product and for this reason, we will talk only about the last one in the corresponding sections.

II. DEPARTMENTS I AND II

Each industry i produces a certain quantity c_i of consumption goods and another quantity $1 - c_i$ of means of production. Therefore, we can represent its activity by means of two equations, one for each part of it, establishing the following equation systems:

$$\Sigma_i(1 - c_i)a_{ij}p_i(1 + r) + (1 - c_i)l_j = (1 - c_i)p_j \qquad j = 1, 2, \ldots, n$$

$$\Sigma_i c_i a_{ij} p_i(1 + r) + c_i l_j = c_i p_j \qquad j = 1, 2, \ldots, n$$

The first system represents the production of capital goods and the second one that of consumption goods, named by Marx (1992:47) respectively departments I and II of production. If no industry belongs to the two departments either $c_i = 0$ or $c_i = 1$ for each i, naturally in this cases $c_i = 0$ for at least one i and $n > 1$. Adding up the equations in each system yields, respectively, the following equations $\Sigma_j \Sigma_i(1 - c_i)a_{ij}p_i(1 + r) + \Sigma_j(1 - c_i)l_j = \Sigma_j(1 - c_i)p_j$ and $\Sigma_j \Sigma_i c_i a_{ij} p_i(1 + r) + \Sigma_j c_i l_j = \Sigma_j c_i p_j$. To simplify, I will introduce the variables $p_I = \Sigma_j(1 - c_i)p_j$, $c_I = \Sigma_j \Sigma_i(1 - c_i)a_{ij}p_i/p_I$, $l_I = \Sigma_j(1 - c_i)l_j$, $p_{II} = \Sigma_j c_i p_j$ and $l_{II} = 1 - l_I$. The quotient c_I indicates the fraction of the production of department I consumed in the same department, as the rest of its production is consumed in department II and the total labor is employed in the two departments, we can write the preceding equation systems under the following forms:

$$c_1 p_1 (1 + r) + l_1 = p_1 \tag{5}$$

$$(1 - c_1) \, p_1 (1 + r) + 1 - l_1 = p_{II} \tag{6}$$

Using this notation, the capital stock may be represented by means of the quotient p_1/p_{II}. Substituting the denominator in this fraction on the left-hand side of (6) gives $p_1/[(1 - c_1)p_1(1 + r) + 1 - l_1] = 1/[(1 - c_1)(1 + r) + (1 - l_1)/p_1]$. On the other hand, solving (5) yields $p_1 = l_1/[1 - c_1(1 + r)]$ and substituting p_1 according to this result on the right-hand side of the preceding equation, we get $1/\{(1 - c_1)(1 + r) + (1 - l_1)[1 - c_1(1 + r)]/l_1\} = l_1/\{l_1(1 - c_1)(1 + r) + (1 - l_1)[1 - c_1(1 + r)]\} = l_1/\{l_1 + l_1 r - c_1 l_1 (1 + r) + 1 - c_1(1 + r) - l_1 + l_1 c_1(1 + r)]\}$. Simplifying the last expression, we obtain:

$$KS = l_1/[1 - c_1(1 + r) + l_1 r] \tag{7}$$

This formula shows that, given r and l_1, the capital stock depends only on c_1. For this reason, in the next section we will study in more detail how this variable is determined.

III. THE HOMOTHETIC AND COMPLEMENTARY SECTORS

Let $A = [a_{ij}]$, $A^* = [a_{ji}]$ and λ_A represent the Frobenius root of A. From the assumptions adopted, it follows that $0 < \lambda_A < 1$, as shown in Benítez (2009). Therefore, according to Theorem 7 in Chapter 13 of Gantmatcher (1966), there is at least one vector $q \geq 0$, $q \neq 0$ such that $A^* q = \lambda_A q$. As the magnitude of q may be fixed arbitrarily, we will assume, for reasons indicated below, that a q has been chosen satisfying the following conditions: a) the magnitude of q is such that $\min\{1 - c_i - q_i = 0 \mid i = 1, 2, ..., N\}$ and b) q is one of the vectors maximizing the sum $\sum_j q_j l_j$. After q has been determined, multiplying each equation j of (1) by the corresponding factor q_j results in the following system:

$$\sum_i q_j a_{ij} p_i (1 + r) + q_j l_j = q_j p_j \quad j = 1, 2, ..., n \tag{8}$$

This system is homothetic, a condition characterized by the fact that the proportion between the quantity of each good used as a means of production and the quantity produced of the same good is the same for all the goods, implying that KS is constant. Indeed, in (8) this proportion is equal to $\sum_j \sum_i q_j a_{ij} p_i / \sum_j q_j p_j$ permuting the indexes i and j in the numerator, we can express the function as

$\Sigma_i\Sigma_j q_i a_{ji} p_j/\Sigma_j q_j p_j = \Sigma_j\Sigma_i q_i a_{ji} p_j/\Sigma_j q_j p_j$. Because $A^*q = \lambda_A q$, for each j, we have $\Sigma_i q_i a_{ji} = \lambda_A q_j$ and substituting in the preceding quotient, we get $\Sigma_i\Sigma_j q_i a_{ji} p_j/\Sigma_j q_j p_j = \Sigma_j \lambda_A q_j p_j/\Sigma_j q_j p_j$. Therefore:

$$\Sigma_j\Sigma_i q_j a_{ij} p_i/\Sigma_j q_j p_j = \lambda_A \quad \forall \quad r \in [0, R] \tag{9}$$

On the other hand, the condition a) assumed to define q implies that system (8) is embedded in department I and, together with b) assures that, with regard to the quantity of labor employed, (8) is the greater homothetic system in this situation. As shown by Benítez (1986), there may be other homothetic systems, but for the purposes of this paper it is enough with the one satisfying conditions a) and b), to which I will refer as the homothetic sector.

According to the preceding definitions, for each j the part of the production of the jth industry included in department I may be divided into two fractions: the one integrating the homothetic sector, equal to q_i, and the rest equal to $1 - c_i - q_i$. A corresponding representation of department I by means of two equation systems can be formulated, (8) and the following system, to which I will refer as the complementary sector:

$$\Sigma_i(1 - c_i - q_i)a_{ij}p_i(1 + r) + (1 - c_i - q_i)l_j =$$

$$(1 - c_i - q_i)p_j \qquad j = 1, 2, \dots, n \tag{10}$$

Therefore, this system is the complement of the homothetic system in department I. Despite the fact of being a residual determined by the two other systems just mentioned above, the complementary sector is relevant to our study, as will be shown. Summing up the equations of systems (8) and (10) result, respectively, in $\Sigma_j\Sigma_i q_i a_{ij} p_i(1 + r) + \Sigma_j q_i l_j = \Sigma_j q_i p_j$ and $\Sigma_j\Sigma_i(1 - c_i - q_i)a_{ij}p_i(1 + r) + \Sigma_j(1 - c_i - q_i)l_j = \Sigma_j(1 - c_i - q_i)p_j$. To simplify, I will introduce the variables $p_h = \Sigma_j q_i p_j$, $l_h = \Sigma_j q_j l_j$, $p_c = \Sigma_j(1 - c_i - q_i)p_j$ and $c_c = \Sigma_j\Sigma_i(1 - c_i - q_i)a_{ij}p_i/p_c$. As the quantity of labor employed in (10) is $l_I - l_h$, using this notation and (9), department I may be represented by means of the following system:

$$\lambda_A p_h(1 + r) + l_h = p_h$$

$$c_c p_c(1 + r) + l_I - l_h = p_c \tag{11}$$

Solving each of these equations for the corresponding price yields $p_h = l_h/[1 - \lambda_A(1 + r)]$ and $p_c = (l_I - l_h)/[1 - c_c(1 + r)]$. On the other hand, it follows from system (11) that $c_I = (\lambda_A p_h + c_c p_c)/(p_h + p_c)$. Substituting prices in this equation for their equivalence, we get $c_I = \{l_h \lambda_A/[1 - \lambda_A(1+r)] + (l_I - l_h)c_c/[1 - c_c(1+r)]\}/\{l_h/[1 - \lambda_A(1+r)] + (l_I - l_h)/[1 - c_c(1+r)]\}$ and multiplying both the numerator and the denominator in this formula by $[1 - \lambda_A(1 + r)][1 - c_c(1 + r)]$ yields:

$$c_I = \{l_h \lambda_A[1 - c_c(1 + r)] + (l_I - l_h) c_c[1 - \lambda_A(1 + r)]\}$$

$$/\{l_h[1 - c_c(1 + r)] + (l_I - l_h)[1 - \lambda_A(1 + r)]\} \qquad (12)$$

This formula permits one to observe that $c_I \in [\lambda_A, c_c]$ whenever $\lambda_A \neq c_c$ and also that given r, λ_A, l_h and l_I, c_I depends only on c_c, a variable that will be studied with more detail in the next sections. On the other hand, the preceding divisions permit one to define the vector $l = (l_h, l_c, l_{II})$, which is unique for each system of type (1). The values of l are contained in the set $S(l) = \{l \in R^3 \mid (l_h, l_c, l_{II}) \geq 0$ and $l_h + l_c + l_{II} = 1\}$ and may vary widely. Notwithstanding, if (1) is homothetic, then $l_I = R/(1 + R)$, $l_{II} = 1/(1 + R)$ and, as in this case department I is also homothetic, the distribution of labor is given by:

$$l_R = (R/(1 + R)^2, 1/(1 + R)^2, 1/(1 + R)) \qquad (13)$$

## IV. A LOWER BOUND FOR THE CAPITAL STOCK

To study the capital stock, the following proposition will be useful.

Lemma 1.II Let KS_c be the proportion capital/(net product) in the complementary sector, $KS(R)$ and $KS_c(R)$, respectively, the limits of $KS(r)$ and $KS_c(r)$ when r tends to R from below. Then:

a) $KS(r) < 1/r \ \forall \ r \in [0, R]$ b) $KS(R) = 1/R$

$\qquad (14)$

c) $KS_c(r) < 1/r \ \forall \ r \in [0, R]$ d) $KS_c(R) \leq 1/R$

The graph of $KS(R)$ as a function of R is the hyperbole equilateral shown in Figure 1.

Figure 1. *Upper and lower bounds for the capital stock when R = 1, m = 1 and l = l$_R$*
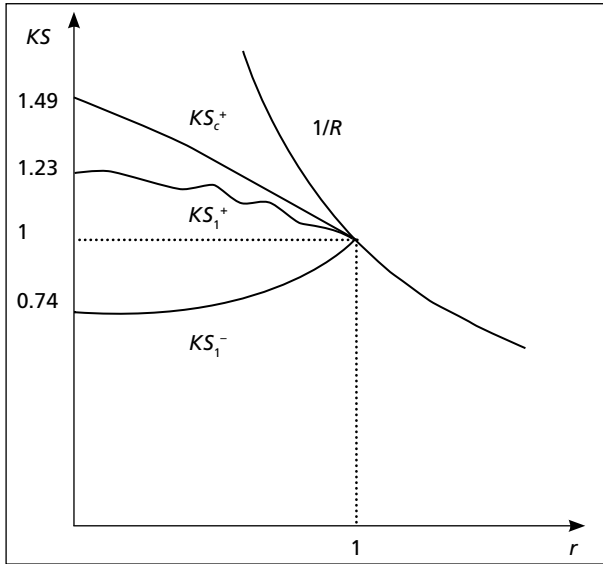
Let $G_{KS}(R) = \{(r, KS) \mid 0 \leq r < R$ and $0 < KS < 1/r\} \cup \{(R,1/R)\}$, according to (14.a) and (14.b) this set contains the graphs of $KS(r)$ of all the systems (1) that share the same R. A natural question investigating the upper and lower bounds for KS is to ask if, given R, there are some other limits for KS apart from the hyperbole just mentioned and the horizontal axis, a question answered negatively in the following proposition.

Theorem 2.III Given R, $\forall (r, T) \in G_{KS}(R)$, there is at least one system of type (1) for which $KS(r) = T$.

Accordingly, the ordinate of each point of $G_{KS}(R)$ is equal to the capital stock of at least one system of type (1) for the corresponding r. Therefore, given any R the magnitude of capital may be as close to zero as desired $\forall r \in [0, R]$. Nevertheless, as it will be shown in this section, the distribution of labor among the departments and sectors of production determines a nonzero lower bound for the systems of type (1) sharing a given R. To show this, the following proposition will be useful, valid if $l_I - l_h > 0$.

Lemma 2.IV c_I is a monotonous increasing function of c_c.

It follows from this result that, given R and l, the smallest capital corresponds to the system of type (1) where c_c is the smallest. Because in each industry at least one produced good is employed, this quantity is always greater than zero, but I will make it equal to zero in (12) to calculate the limit (c_1^-) of the proportion c_1 when c_c tends to zero. After simplifying, one gets $c_1^- = \lambda_A/\{1 + (l_1/l_h - 1)[1 - \lambda_A(1 + r)]\}$. On the other hand, the first equation of (11) may be written in the form $\lambda_A(1 + r) + l_h/p_h = 1$. The second term, on the left-hand side tends to zero when r tends to R (see A.II in "Appendix") so that we have $\lambda_A(1 + R) = 1$ and:

$$\lambda_A = 1/(1 + R) \tag{15}$$

Substituting in the previous expression gives $c_1^- = 1/(1 + R)/\{1 + (l_1/l_h - 1)[1 - (1 + r)/(1 + R)]\}$ and multiplying everything by $(1 + R)$, we obtain after simplifying:

$$c_1^- = 1/[(1 + R) + (l_1/l_h - 1)(R - r)] \tag{16}$$

According to (7), KS is a monotonous increasing function of c_1. For this reason, to obtain a lower bound for the capital stock (KS^-), it is enough to substitute this value of c_1 in (7) resulting in:

$$KS^- = l_1/[1 - c_1^-(1 + r) + l_1 r] \tag{17}$$

We may add that $\forall \in [0, R]$, this formula determines the greater lower bound for KS given R and l. This is because for every $T > KS^-$ it is possible to choose c_c small enough for the resulting system to verify $KS < T$. Figure 1 presents an example of a lower bound for KS.

V. A FAMILY OF SUBSETS

For each R, let $C(R)$ be the set of all the systems of type (1) that share the same R value. An important property of this set is presented next.

Proposition 1.V Given R and l, $\forall$ $(r; T) \in G_{KS}(R)$, there is at least one system of type (1) for which $KS(r) > T$.

This proposition implies that, given R, the vector l does not impose any upper limit on the capital stock. For this reason, in this section we are going to

define a family of subsets of $C(R)$, whose complement in $C(R)$ may be reduced as much as desired and such that in each one of them a given vector l determines a nontrivial upper bound for the capital stock, as will be shown in the next section.

According to (15) (12) and (7), given l and R the capital stock depends only on c_c, so that we will pay particular attention to this sector. It is convenient to observe that, if the capital stock is constant, $KS_c(r) \leq 1/R \; \forall \; r \in [0, R]$ according to (14.d). Then, if $KS_c(r*) > 1/R$ for a certain $r* \in [0, R]$, necessarily KS_c diminishes when r changes from $r*$ to R. From equation (A.1) (see "Appendix"), it follows that when $r = r*$, the wage in the complementary sector, measured with the net product of the sector, is equal to $[1 - KS_c(r*)r*]$. Let m be the reduction of the capital stock taking place when r changes from $r*$ to R measured with the wage in $r*$, then $m = [KS_c(r*) - 1/R]/[1 - KS_c(r*)r*]$. Therefore, if for every $r \in [0, R]$, the capital stock diminishes in an amount equivalent to m times the wage in r when the rate of profit changes from r to R, we have $KS_c(r) = m[1 - KS_c(r)r] + 1/R$ and so:

$$KS_c(r) = (1/R + m)/(1 + mr) \; \forall \; r \in [0, R] \tag{18}$$

For each couple (m, R) where $R > 0$ and $m \geq 0$, let $C_m(R)$ be the set of all the systems of type (1) for which $KS_c(r)$ is less than the value determined by (18) $\forall \; r \in [0, R]$. We may ask if for every $m \geq 0$ the amount of capital determined by (18) corresponds to at least one system of type (1) and also if for every system of type (1) there is an $m \geq 0$ such that the capital stock is less than the value determined by (18) for every $r \in [0, R]$. Both questions receive an affirmative answer based on the following proposition.

Lemma 3.VI The family of functions (18) possesses the following properties: a) $\forall \; T \in [1/R, 1/r[$ and $\forall \; r \in [0, R]$ there is an m such that $KS(r) = T$, b) for every $m > 0$ the graph of (18) is below the hyperbole $1/R$ and c) for every $r \in [0, R]$, $KS(r)$ is a monotonous increasing function of m.

It is convenient to observe that according to (14.d) for every $r* \in [0, R]$, when r changes from $r*$ to R, the wage is reduced to a quantity equal to or greater than zero implying that the simultaneous increase in profits is smaller than or equal to the wage in $r*$. Consequently, in the systems verifying (18), the capital stock in the complementary sector diminishes with this change an amount equivalent to at least m times the increase in profits. For this reason, $C_m(R)$ may be described as the set integrated by all the elements of $C(R)$ except a part of

those that, for some $r* \in [0, R]$, when r changes from $r*$ to R, the capital stock in the complementary sector diminishes in an amount equal to at least m times the simultaneous increase in profits. It follows from Lemma 3 that when m increases starting from zero the set $C_m(R) \cap C(R)$ grows continually from $C_0(R)$, a set that includes only those systems where capital always grows when r changes from an $r* \in [0, R]$ to R, and tend to be equal to $C_m(R)$ when m tends to infinity. Finally, it is worth noting that a system of type (1) pertaining to a set $C_m(R)$ satisfies a restriction regarding the relative price of the particular sets of goods integrated by the capital stock and the net product of the complementary sector but there is no other restriction on relative prices.

VI. AN UPPER BOUND FOR THE CAPITAL STOCK

For each m and for each $r < R$, (18) permits one to establish an upper bound $c_c^+(r)$ for $c_c(r)$ valid for all the production systems pertaining to $C_m(R)$. Indeed, $KS_c(r) = c_c(r)/[1 - c_c(r)]$; it follows from this equation and (18) that an upper bound for $c_c(r)$ verifies $c_c^+(r)/[1 - c_c^+(r)] = (1/R + m)/(1 + mr)$ implying that $c_c^+(1 + mr) = (1 - c_c^+)(1/R + m)$, hence $c_c^+[(1 + mr) + (1/R + m)] = (m + 1/R)$ and:

$$c_c^+ = (1/R + m)/[1 + m(1 + r) + 1/R] \tag{19}$$

On its turn, this result permits one to establish the following upper bound for the capital stock:

$$KS(r)_m^+ = l_1/[1 - c_1(c_c^+)(1 + r) + l_1 r] \tag{20}$$

obtained by substituting first c_c for c_c^+ in (12) and the result $(c_1(c_c^+))$ substitutes c_1 in (7).

To employ this formula, which calculates an upper bound for the capital stock of a particular system of type (1), an assumption is made about the smallest set $C_m(R)$ containing the system, something that may depend on empirical or theoretical considerations. In this regard, the set $C_1(R)$ has a special interest for the following reasons: a) it includes all the production systems where $KS_c(r)$ is a constant or a monotonous increasing function, b) it includes a part of those systems where $KS_c(r)$ is monotonously decreasing or not a monotonous function and c) it excludes only a part of those systems where for some $r* \in [0, R]$, when r changes from $r*$ to R, $KS_c(r)$ diminishes an amount at least equal to the

corresponding increase in profits. It is to remark that in such cases the sum of capital plus profits, measured with the net product, either keeps constant or decreases. Formulas (17) and (20) permit one to estimate the capital stock as the average of the values determined by both, to which it corresponds a maximum error equal to one half of the difference between them. After simplifying, the following formula is established:

$$E_m KS(r) = (KS^+ + KS_m^-)/2 \pm (KS^+ - KS_m^-)/2 \qquad (21)$$

The value determined by (12) for $(c_1(c_c^+))$ is always equal to $1/R$ when $r = R$. For this reason, the error in (21) is equal to zero in this case, a result indicating that it may be as small as desired for values of r sufficiently near to R. On the other hand, normally the error will be greater when r approaches zero. Figure 1 presents the graphs corresponding to (19) and (20) when $m = 1$, $R = 1$ and $l = (1/4, 1/4, 1/2)$ is determined by (13).

VII. Upper and lower bounds for wages

Let $G_w(R) = \{(r, w) \mid 0 < r < R \text{ and } 0 < w < 1\} \cup \{(0, 1), (R, 0)\}$ this set contains all the possible graphs of $w(r)$ and is equal to the interior points of a rectangle as those shown in Figures 2 and 3 plus the extreme points of $S(R)$. We may ask if every point of this set pertains to the wage-profit curve of at least one system of type (1), an affirmative answer is given by the next proposition.

Figure 2. *Upper and lower bounds for wages when R = 1, m = 1, t = 0 and l = l_R*

Figure 3. *Upper and lower bounds for wages when R = 1, m = 1, t = 1 and l = l_R*
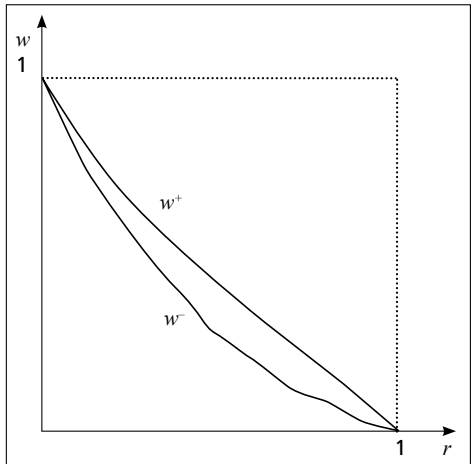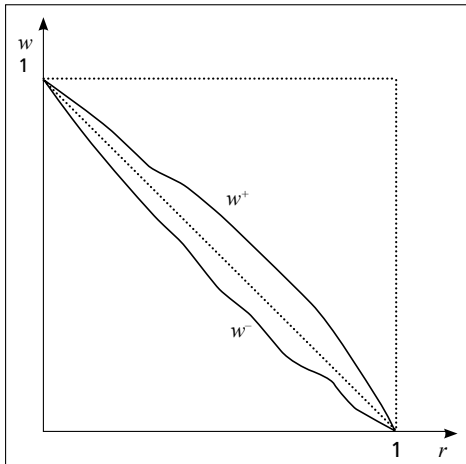
Theorem 3.VII Given R, $\forall\; r \in [0, R]$ and for every F such that $0 < F < 1$, there is at least one system of type (1) for which $w(r) = F$.

Then, given R and r such that $0 < r < R$, $w(r)$ may have any value comprised between 0 and 1. This result together with the fact that the wage-profit curve is monotonously decreasing imply that the curve adopts rather curious forms for certain systems of type (1). For instance, choosing $(r, w(r))$ close enough to $(0, 0)$, the theorem guaranties that there is a system whose wage-profit curve will look like the square formed by the two axis while choosing $(r, w(r))$ sufficiently close to $(R, 1)$ the corresponding curve will look like the square formed by two lines parallel to the axis intersecting in this point. Nevertheless, a given vector l permits one to limit this diversity establishing an upper bound $w(r)^{+}$ and, for each m, a lower bound $w(r)_{m}^{-}$, by substituting respectively KS^{-} and KS_{m}^{+} in (4.a). In this manner, we obtain after solving for $w(r)$:

a) $w(r)^{+} = (1 - KS^{-}r)/(1 + tr)$ and

b) $w(r)_{m}^{-} = (1 - KS_{m}^{+}r)/(1 + tr)$ \hfill (22)

On their turn, these bounds permit one to estimate the wage as their average value with a maximum error equal to one half of the difference between them. After simplifying, we arrive at the following formula:

$$E_{m}w(r) = [2 - (KS^{+} + KS_{m}^{-})r]/2(1 + tr)$$

$$\pm\, (KS^{+} - KS_{m}^{-})r/2(1 + tr) \hfill (23)$$

It is convenient to note that the error is zero when $r = 0$ or $r = R$. Therefore, it may be as small as desired if the levels of r being considered are close enough to any of these values. Moreover, the error diminishes as t increases. Figures 2 and 3 show the graphs of the last three formulas, respectively, when $t = 0$ and $t = 1$, with $R = 1$ and vector $l = (1/4, 1/4, 1/2)$ is determined according to (13).

FINAL REMARKS

Assuming that $t = 0$, it follows from equation (4.a) that when the capital stock is constant, w is a linear function of r and its graph is the straight segment $S(R)$.

Loosely speaking, we may refer to the fact that w is a continuous function of KS saying that the wage-profit curve keeps close to a straight line if the capital stock maintains close to a constant value. In the economic literature, two sufficient conditions have been pointed out for the value of capital to be constant: a) the equality between relative prices and relative values and b) the homothetic condition of the production system. In this paper, we arrived at the definition of pairs of conditions such that given any one of them the other one is sufficient for the proximity between $w(r)$ and $S(R)$. Indeed, if given a couple (m^*, l^*) a $w(r)$ function either contained in or near to the interval determined by (22) is considered to be close to $S(R)$, we can establish the following conclusion:

Proposition 2. In a system of type (1), in order for the wage-profit curve to be close to $S(R)$, it is not required for the proportion between quantities produced and consumed to be near $1 + R$, neither that relative prices be close to the corresponding value proportions. Any of the following two conditions is sufficient if the other one is satisfied: α) that the system belongs to a set $C_m(R)$ such that m is either close to or less than m^* and β) that the vector l of the system is close to l^*.

In relation to the value of m, the empirical relevance of the set $C_l(R)$ for which some arguments were offered in sixth section may be confirmed by the quasilinearity of the wage-profit curves reported by several empirical researches, as those by Ochoa (1989), Michl (1991), Shaikh (1997), and Tsoulfidis and Rieu (2006). Indeed, if –as argued in some of these works– the studied economies satisfy either condition a) or b) indicated above, then they belong to $C_1(R)$. On the other hand, as may be appreciated in Figures 2 and 3, if a production system belongs to $C_1(R)$ the wage-profit curve is close to $S(R)$ if vector l is close enough to l_R. Nevertheless, to verify that this last condition is satisfied in the cases just cited, empirical considerations must be developed requiring another article.

We have chosen l_R to calculate the numerical examples because of its peculiar properties, particularly the relation to segment $S(R)$ that has just been mentioned, but in order to illustrate the existence of the upper and lower bounds studied here any $l \in S(l)$ could have been chosen. It also permitted to illustrate the fact that these bounds can be very close to each other, suggesting that the corresponding estimation of the functions may be used eventually in empirical researches. For instance, the estimation of w by means of (22) in the case of the example presented on seventh section yields an error smaller than 6.1% and

6.13% for any $r < 20\%$, considering respectively $t = 0$ and $t = 1$, and the error decreases tending to zero as r tends to zero.

Finally, we may add that the approach followed here to study functions $KS(r)$ and $w(r)$, largely based on the contributions of the authors mentioned in the introduction, is justified by the results established. Naturally, our research covered only a few properties of these functions and in order to reach further advances, for instance concerning the degree in which changes in l affect the upper and lower bounds of the functions, more studies based either on this or in other perspectives are required.

APPENDIX

A.I See Benítez (2009).

A.II Proposition (3.a) and b) from Theorem 1 imply that $w > 0 \ \forall \ r \in [0, R]$, this result and (4.a) imply that $KS(r)r < 1$ and a) follows. According to (4.a), when r increases from 0 to R we have $\lim_{r \to R} KS(r)r + \lim_{r \to R} w(1 + tr) = 1$. When $r = R$ the previous equation may be written as $K(R)R + \lim_{r \to R} w(1 + tr) = 1$. As shown in Benítez (2009), $\lim_{r \to R} w = 0$ thus, substituting in the preceding equation yields $KS(R)R = 1$ implying b). On the other hand, the relation between the cost and the price of production in the complementary sector obeys the relation $C_c(r)(1 + r) + l_c = C_c(r) + PN_c$ where C_c, l_c are respectively the cost of the means of production and the wages paid while PN_c is the difference between the value of production and its cost (net product of the sector). Consequently, $C_c(r)r + l_c = PN_c \Rightarrow [C_c(r)/PN_c]r + l_c/PN_c = 1$, an equation that may be written as:

$$KS_c(r)r + l_c/PN_c = 1 \tag{A.1}$$

Because $l_c/PN_c > 0 \ \forall \ r \in [0, R]$, it follows that $KS_c(r)r < 1 \Rightarrow$ c). Finally, as $l_c/PN_c \geq 0$ when $r = R$, we have $KS_c(R)R \leq 1 \Rightarrow$ d).

A.III We will prove successively that there is a system of type (1) where the capital stock is: a) smaller than T, b) greater than T and c) equal to T.

a) Consider the following system:

$$a_{11}p_1(1 + r) + l_1 = p_1 \tag{A.2}$$

$$a_{22}p_2(1 + r) + l_2 = p_2$$

Solving the equations, we obtain $p_1 = l_1/[1 - a_{11}(1 + r)]$ and $p_2 = l_2/[1 - a_{22}(1 + r)]$ so that $p_1/p_2 = (l_1/l_2)\{[1 - a_{22}(1 + r)]/[1 - a_{11}(1 + r)]$ while $KS = [a_{11}p_1 + a_{22})p_2]/[(1 - a_{11})p_1 + (1 - a_{22})p_2] = [a_{11}p_1/p_2 + a_{22}]/[(1 - a_{11})p_1/p_2 + (1 - a_{22})]$. Let us choose $a_{11} = 1/(1 + R)$ and a_{22} such that $a_{22}/(1 - a_{22}) = \min\{T/2, 1/2(1 + R)\}$. Then, in order for KS to be lesser than T, it suffices to choose l_1 and l_2 such that, for the given r, the proportion p_1/p_2 is close enough to zero.

b) Given R, and $r \in [0, R]$, let a be such that $a = 1/(1 + R)$ and let $l_0 > 0$ be a given quantity of labor. Consider the following spring-like production system:

$$ap_1(1 + r) + l_0/n = p_1 \qquad\qquad\qquad\qquad\qquad (A.3)$$

$$(1 - a)p_{j-1}(1 + r) + ap_j(1 + r) + l_0/n = p_j \qquad\qquad j = 2, 3, \ldots, n$$

In this system, a units produced by each branch j are employed in its own production and the remaining $(1 - a)$ units in the production of the $(j + 1)$th branch, except for the nth branch where the corresponding $(1 - a)$ units constitute the net product of the system. Solving for the price of the good produced in each branch yields:

$$p_1 = (l_0/n) / [1 - a(1 + r)]$$

$$p_j = [l_0/n + (1 - a)p_{j-1}(1 + r)]/[1 - a(1 + r)] \qquad\qquad j = 2, \ldots, n$$

This system permits one to observe that R is determined by the equation $1 - a(1 + R) = 0$, implying that $a = 1/(1 + R)$ as initially established. Defining $b = (1 - a)(1 + r)/[1 - a(1 + r)]$ we can write the previous system as:

$$p_1 = (l_0/n)/[1 - a(1 + r)]$$

$$p_j = p_1 + bp_{j-1} \qquad\qquad\qquad\qquad\qquad\qquad j = 2, \ldots, n$$

Substituting p_{j-1} for p_1 in the second equation we obtain $p_2 = p_1 + bp_1$. Then, substituting p_{j-1} for $p_1 + bp_1$ in the third equation results in $p_3 = p_1 + b[p_1 + bp_1]$, and continuing in this manner, we get $p_j = p_1 + b[p_1 + bp_1 + \ldots + b^{j-2}p_1]$ for every $j > 1$. Consequently, the preceding system may be written as:

$$p_1 = (l_0 / n) / [1 - a (1 + r)] \tag{A.4}$$

$$p_j = p_1(1 + b + b^2 + \ldots + b^{j-1}) \qquad j = 2, \ldots, n$$

On the other hand, the capital stock in (A.3) is equal to $[p_1 + p_2 + \ldots + p_{n-1} + ap_n]/(1 - a)p_n$, a quotient that may be written in either of the following forms:

a) $[p_1 + \ldots + p_{n-1} + ap_n]/(1 - a)p_n$

b) $[1/(1 - a)][p_1 + \ldots + p_{n-1}]/p_n + a/(1 - a) \tag{A.5}$

On its turn, by substituting each price except p_1 in the quotient $[p_1 + p_2 + \ldots + p_{n-1}]/p_n$ for its equivalent in (A.4) we obtain, after simplifying:

$p_1[1 + (1 + b) + (1 + b + b^2) + \ldots + (1 + b + b^2 \ldots + b^{n-2})]/p_1(1 + b + b^2 \ldots + b^{n-1})$

$= [1 + (1 + b) + (1 + b + b^2) + \ldots + (1 + b + b^2 \ldots + b^{n-2})]/(1 + b + b^2 \ldots + b^{n-1})$

$= [1 + (1 + b) + (1 - b^3)/(1 - b) + \ldots + (1 - b^{n-1})/(1 - b)]/[(1 - b^n)/(1 - b)]$

Multiplying the numerator and denominator by $(1 - b)$ results in $[(1 - b) + (1 - b^2) + (1 - b^3) + \ldots + (1 - b^{n-1})]/(1 - b^n) = [(n - 1) - (b + b^2 + b^3 + \ldots + b^{n-1})]/(1 - b^n) = [(n - 1) - (1 - b^n)/(1 - b)]/(1 - b^n) = [(n - 1)/(1 - b^n) - 1/(1 - b) = 1/(b - 1) - (n - 1)/(b^n - 1)$. Thus, substituting in (A.5.b) yields:

$$KS = [1/(1 - a)][1/(b - 1) - (n - 1)/(b^n - 1)] + a/(1 - a) \tag{A.6}$$

Because $b > 1 \; \forall \; r \in [0, R]$, the quotient $(n - 1)/(b^n - 1)$ is a monotonous decreasing function of b whose value tends to zero when n tends to infinity. It is important to note that in this case the capital tends to $1/r$. Indeed, $\lim KS_{n \to \infty} = [1/(1 - a)][1/(b - 1)] + a/(1 - a)$. Or, $b - 1 = (1 - a)(1 + r)/[1 - a(1 + r)] - 1 = \{(1 - a)(1 + r) - [1 - a(1 + r)]\} / [1 - a(1 + r)] = r/[1 - a(1 + r)]$, substituting in the preceding expression yields:

$$\lim KS_{n \to \infty} = [1/(1 - a)][1 - a(1 + r)]/r + a/(1 - a)$$

$$= (1/r)\{[1/(1 - a)][1 - a(1 + r)] + ar/(1 - a)\}$$

$$= [1/r(1 - a)]\{[1 - a(1 + r)] + ar\}$$

$$= [1/r(1-a)][1-a] = 1/r \tag{A.7}$$

This is the value of *KS* when all the income corresponds to profits (because $(\lim KS_{n\to\infty})r = 1$). Therefore, for any $T < 1/r$, we can choose *n* great enough in order for the capital of system (A.3) to be greater than *T*.

    c) Consider the production system integrated by the two equations of (A.2), each one of them multiplied by $(1-u)$ where $u \in [0, 1]$ and the *n* equations of (A.3) each one of them multiplied by *u*. The capital stock in the aggregated system is determined by:

$$\{(1-u)[a_{11}\, p_1 + a_{22}\, p_2] + u[p_1 + p_2 + \ldots + p_{n-1} + ap_n]\}/$$

$$\{u[(1-a_{11})\, p_1 + (1-a_{22})\, p_2] + (1-u)(1-a)\, p_n\}$$

As this quotient is a continuous function of *u*, it adopts all the values between those corresponding to the original systems verifying a) and b), so that there is a *u* verifying c).

A.IV Summing up the two equations of (11) we obtain $(\lambda_A p_h + c_c p_c)(1+r) + l_1 = p_h + p_c$ and dividing this equation by $p_h + p_c$ results $(\lambda_A p_h + c_c p_c)(1+r)/(p_h + p_c) + l_1/(p_h + p_c) = 1$ which may be written as $c_1(1+r) + l_1/(p_h + p_c) = 1$. Solving each equation of (11) for the corresponding price gives $p_h = l_h/[1 - \lambda_A(1+r)]$ and $p_c = (l_1 - l_h)/[1 - c_c(1+r)]$ substituting the prices in the preceding equation yields $c_1(1+r) + l_1/\{l_h/[1 - \lambda_A(1+r)] + (l_1 - l_h)/[1 - c_c(1+r)]\} = 1$ permitting one to verify the lemma because the second term from the left-hand side is a monotonous decreasing function of c_c.

  A.V We define λ_A according to (15) for the given *R*, and we choose as the homothetic sector the first equation in system (11). Then, we define l_h and *l* such that $0 < l_h < l < 1$, $l_0 = l - l_h$ and *a* such that $0 < a < \lambda_A$, the resulting system (A.3) will be the complementary sector. Choosing as sector II the equation $(1 - \lambda_A)\, p_h(1+r) + (1-a)p_n(1+r) + 1 - l = p_{II}$ we have a system respecting the conditions. The capital stock of the complementary sector, measured with the net product of that sector is determined by (A.5). This conclusion and (A.7) imply that $\lim_{v\to\infty}[c_c/(1-c_c)] = 1/r$, hence $(\lim_{n\to\infty} c_c)/[1 - (\lim_{n\to\infty} c_c)] = 1/r \Rightarrow (\lim_{n\to\infty} c_c)r = 1 - (\lim_{n\to\infty} c_c) \Rightarrow (\lim_{n\to\infty} c_c)(1+r) = 1$. From this result and (12), it follows that $\lim_{n\to\infty} c_I = \lim_{n\to\infty} c_c$, which together with the equation $(\lim_{n\to\infty} c_c)(1+r) = 1$ and (7) imply that $\lim_{n\to\infty} KS = l/lr = 1/r$.

A.VI Making $m = (T – 1/R)/[1 – Tr]$ verifies a). From (18.b), it follows that $KS(r)$ $(1 + mr) = m + 1/R$ so that $KS(r) = (m + 1/R)/(1 + mr)$, multiplying the two sides by r, we conclude that $KS(r)r = (mr + r/R)/(1 + mr) < 1$ if $r < R$, so that $KS(r) < 1/r \ \forall \ r \in [0, R]$. Finally, deriving $(m + 1/R)/(1 + mr)$ as a function of m results in $[1 + mr – r(m + 1/R)]/(1 + mr)^2 = (1 – r/R)/(1 + mr)^2 > 0 \ \forall \ r \in [0, R]$.

A.VII Let $T = (1 – F)/r$, according to Theorem 2 there is a system such that $KS(r) = T$. It follows from (4.a) that in this system we have $w(r) = 1 – KSr = 1 – [(1 – F)/r]r = F$.

A.VIII Some data of the graphs in Figures 1, 2 and 3.

Table A.1*

r	KS_1^-	KS_1^+	$w^+(0)$	$w^+(1)$	$w^-(0)$	$w^-(1)$	KS_C^+	$(r, 1/r)$
0	0.749	1.23	1	1	1	1	1.499	(0.6, 1.66)
0.1	0.745	1.24	0.925	0.843	0.876	0.796	1.468	(0.7, 1.42)
0.2	0.744	1.23	0.851	0.709	0.752	0.627	1.428	(0.8, 1.25)
0.3	0.747	1.20	0.775	0.596	0.628	0.490	1.380	(0.9, 1.11)
0.4	0.755	1.18	0.697	0.498	0.528	0.377	1.327	(1, 1)
0.5	0.769	1.19	0.615	0.41	0.403	0.269	1.272	(1.1, 0.9)
0.6	0.789	1.13	0.526	0.329	0.322	0.201	1.215	(1.2, 0.83)
0.7	0.818	1.13	0.427	0.251	0.207	0.122	1.159	(1.3, 0.76)
0.8	0.859	1.06	0.312	0.173	0.152	0.084	1.104	(1.4, 0.71)
0.9	0.916	1.04	0.174	0 .092	0.056	0.029	1.051	(1.5, 0.66)
1	1	1	0	0	0	0	1	(1.6, 0.62)

*(0) and (1) indicate different values of t.

References

Benítez, Alberto (1986), "L'étalon dans la theorie de P. Sraffa", *Cahiers d'économie politique,* 12:131-146. [(1990), "La mercancía patrón en la teoría de Piero Sraffa", *Lecturas de Economía*, 32-33: 45-68].

Benítez, Alberto (1995), *Desequilibrio y precios de producción*, México, UAM-Siglo XXI Editores.

Benítez, Alberto (2009), "El pago del salario", *Investigación Económica,* 270: 69-96. [(2010) "The payment of wages", *Denarius,* 20: 193-219].

Bidard, Christian (2004), *Prices, Reproduction, Scarcity*, Cambridge, Cambridge University Press.

Gantmatcher, Felix (1966), *Matrix Theory*, New York, Chelsea Publishing.

Marx, Karl (1990), *Capital*, Volume 1, New York, Penguin Books.

——— (1992), *Capital*, Volume II, New York, Penguin Books.

Michl, Thomas (1991), "Wage-profit curves in US manufactures", *Cambridge Journal of Economics,* 15: 271-286.

Morishima, Mischio (1973), *Marx's Economics*, Cambridge, Cambridge University Press.

Neuman, Joseph von (1945), "A Model of General Economic Equilibrium", *Review of Economic Studies*, 13: 135-145.

Ochoa, Eduardo (1989), "Values, prices and wage-profit curves in the US economy" *Cambridge Journal of Economics*, 13: 413-429.

Pasinetti, Luigi (1977), *Lectures in the theory of production,* New York, Columbia University Press.

Ricardo, David (2004), *The works and correspondence of David Ricardo*, Indianapolis, Liberty Found, Inc.

Shaikh, Anwar (1997), "The Empirical Strength of the Labor Theory of Value", in Bellofiori, Ricardo (ed), *Marxian Economics: A Reappraisal*, Vol. 2, New York, St. Martin's Press.

Smith, Adam (1981), *An inquiry into the nature and causes of the wealth of nations*, Vol. I, Indianapolis, Liberty Found, Inc.

Sraffa, Piero (1960), *Production of commodities by means of commodities*, Cambridge, Cambridge University Press.

Steedman, Ian (1977), *Marx after Sraffa*, London, New Left Books.

Tsoulfidis, Lefteris, and Rieu, Dong Ming (2006), "Labor Values, Prices of Production, and Wage-Profit Rate Frontiers of the Korean Economy", *Seoul Journal of Economics,* 3: 275-295.